7.3. Operational Properties I.
(Translation Theorems)

7.1, 7.2 : Learn how to transform / inverse transform some basic functions.

7.3, 7.4 : Learn rules about how to transform some combinations of functions.

Thm 1 : (1st translation thm / s-axis)

If $\boxed{\mathcal{L}(f(t)) = F(s) \text{ then } \mathcal{L}(e^{at} f(t)) = F(s-a)}$ for any real number a.

Proof: $\mathcal{L}(e^{at} f(t)) \overset{\text{Def}}{=} \int_0^\infty e^{-st} e^{at} f(t)\, dt$

$$= \underbrace{\int_0^\infty e^{-(s-a)t} f(t)\, dt}_{F(s-a)}$$

Ex: $\mathcal{L}(t^2 e^{3t})$

$a = 3, \quad \mathcal{L}(t^2) = \frac{2!}{s^2} = F(s)$

By Thm 1, $\mathcal{L}(t^2 e^{3t}) = F(s-3) = \frac{2}{(s-3)^2}$

Ex: $\mathcal{L}(e^{-t} \sin(2t))$

$a = -1, \quad \mathcal{L}(\sin(2t)) = \frac{2}{s^2 + 2^2} = F(s)$

By Thm 1, $\mathcal{L}(e^{-t} \sin(2t)) = F(s+1) = \frac{2}{(s+1)^2 + 4}$.

Cor: (1^{st} translation thm for inverse Laplace)

$$\boxed{\text{If } \mathcal{L}^{-1}(F(s)) = f(t) \text{ then } \mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)}$$

Ex: Find $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right)$

Recall: $\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$ (*)

Key point: Realize $\frac{1}{s^2+2s+5}$ is a translation of (*)

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4} \overset{(a=2)}{=} \left(\frac{1}{2}\right)\boxed{\frac{2}{(s+1)^2+4}}$$

$$= F(s+1) \quad \text{for } F(s) = \frac{2}{s^2+2^2}$$

Since $\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin(2t)$

then $\mathcal{L}^{-1}(F(s+1)) = e^{-t}\sin(2t)$

Finally, $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right) = \frac{1}{2} e^{-t}\sin(2t)$.

Ex: $\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+6s+34}\right)$

$$\frac{2s+5}{s^2+6s+34} = \frac{2s+5}{(s+3)^2+25} = 2\cdot\frac{(s+3)}{(s+3)^2+25} - \frac{1}{(s+3)^2+25}$$

$$= 2F_1(s+3) - \frac{1}{5}F_2(s+3)$$

$F_1(s) = \frac{s}{s^2+5^2}$, $F_2(s) = \frac{5}{s^2+5^2}$

$\mathcal{L}^{-1}(F_1(s)) = \cos(5t)$, $\mathcal{L}^{-1}(F_2(s)) = \sin(5t)$

So $\mathcal{L}^{-1}(\ldots) = 2\mathcal{L}^{-1}(F_1(s+3)) - \frac{1}{5}\mathcal{L}^{-1}(F_2(s+3))$

$= 2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t)$

(+) 2nd Translation Thm (t-axis)

<u>Motivation</u>: Consider $f(t) = \begin{cases} e^{-2t} & \text{for } 0 \le t < 1 \\ 0 & \text{for } t > 1. \end{cases}$

Find $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^1 e^{-st}e^{-2t}\,dt = \int_0^1 e^{-(s+2)t}\,dt$

$= \frac{e^{-(s+2)t}}{-(s+2)}\Big|_0^1 = \boxed{\frac{e^{-(s+2)}-1}{-(s+2)}}$ (*).

Original function $f(t)$ is not continuous at $t=1$ but $F(s)$ is continuous

(+) Heaviside function: $u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$

$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t \geq a. \end{cases}$

Rewrite $f(t) = e^{-2t} + \underbrace{u(t-1)}(-e^{-2t}) = \begin{cases} e^{-2t} & t<1 \\ 0 & t \geq 1 \end{cases}$

Thm2 (2nd trans thm) If $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}\left(\underbrace{u(t-a)\, f(t-a)}\right) = e^{-as} F(s) \text{ for any } a>0.$$

Thm 2': $\boxed{\mathcal{L}(u(t-a)\, g(t)) = e^{-as}\, \mathcal{L}(g(t+a))}$

Comparing Thm2 vs Thm 2': $f(t-a) = g(t)$

Ex: $f(t) = e^{-2t} - u(t-1)e^{-2t}$

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{-2t}) - \underbrace{\mathcal{L}(u(t-1)e^{-2t})}$$

$$= \frac{1}{s+2} - e^{-s}\mathcal{L}(e^{-2(t+1)})$$

$$= \frac{1}{s+2} - e^{-s}\mathcal{L}(e^{-2t})\, e^{-2}$$

$$= \frac{1}{s+2} - e^{-(s+2)}\frac{1}{s+2}$$

$$= \frac{1-e^{-(s+2)}}{s+2}$$

$g(t) = e^{-2t}$

(By 2')

$a = 1$

Ex: $f(t) = \begin{cases} f_1(t) & 0 \le t < t_1 \\ f_2(t) & t_1 \le t < t_2 \\ f_3(t) & t \ge t_2 \end{cases}$

Rewrite $f(t) = f_1(t) + u(t-t_1)(f_2(t) - f_1(t)) + u(t-t_2)(f_3(t) - f_2(t))$

$$= f_1 + u(t-t_1)(f_2 - f_1) + u(t-t_2)(f_3 - f_2)$$

$$\boxed{\underline{\text{Cor}}: \quad \mathcal{L}^{-1}\left(e^{-as}F(s)\right) = u(t-a)\,\mathcal{L}^{-1}(F(s))\Big|_{t-a}}$$

$\underline{\text{Ex}}$: $\mathcal{L}^{-1}\left(\dfrac{e^{-s}}{s^2+s}\right) \overset{\text{Cor }(a=1)}{=} u(t-1)\,\mathcal{L}^{-1}\left(\dfrac{1}{s^2+s}\right)\Big|_{t-1}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right) \overset{\text{Partial Fraction}}{=} \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= 1 - e^{-t}$$

Finally $\mathcal{L}^{-1}\left(\dfrac{e^{-s}}{s^2+s}\right) = u(t-1)\left(1 - e^{-(t-1)}\right)$

$\underline{\text{Ex}}$: Solve $y'' - y = \begin{cases} 0 & \text{for } t < 1, \\ 2 & \text{for } t \geq 1 \end{cases}$

$$y(0) = 4, \quad y'(0) = 2.$$

Realize $y'' - y = 2\,u(t-1)$

$$\mathcal{L}(y'' - y) = 2\mathcal{L}(u(t-1))$$

$$s^2 Y - s\,y(0) - y'(0) - Y = 2\frac{e^{-s}}{s} \quad \left(\mathcal{L}(1) = \frac{1}{s}\right)$$

$$(s^2-1)Y - 4s - 2 = 2\frac{e^{-s}}{s}$$

$$Y = \frac{2e^{-s}}{s(s^2-1)} + \frac{4s+2}{s^2-1}$$

$$y = \mathcal{L}^{-1}(Y) = \underbrace{\mathcal{L}^{-1}\left(\frac{2e^{-s}}{s(s^2-1)}\right)}_{①} + \underbrace{\mathcal{L}^{-1}\left(\frac{4s+2}{s^2-1}\right)}_{②}$$

$$① \quad 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s^2-1)}\right) \overset{(a=1)}{=} 2u(t-1)\,\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)\Big|_{t-1}$$

Partial Fraction: $\frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)}$

$$= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$(\ldots)$ $A = -1$, $B = \frac{1}{2}$, $C = \frac{1}{2}$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right) = \mathcal{L}^{-1}\left(-\frac{1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1}\right)$$

$$= -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$$

$$2\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)\Big|_{t-1} = \boxed{-2 + e^{t-1} + e^{-(t-1)}}$$

$$② \quad \mathcal{L}^{-1}\left(\frac{4s+2}{s^2-1}\right) = \mathcal{L}^{-1}\left(\frac{3}{s-1} + \frac{1}{s+1}\right) = 3e^{t} + e^{-t}$$

$$\frac{4s+2}{s^2-1} = \frac{D}{s-1} + \frac{E}{s+1} \quad \ldots\ldots \quad D = 3,\ E = 1.$$

Final answer: $y = u(t-1)\left(-2 + e^{t-1} + e^{1-t}\right)$
$+ 3e^{t} + e^{-t}.$